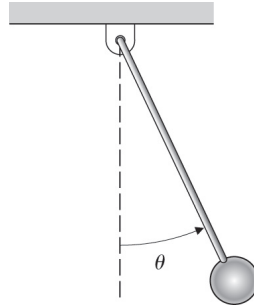


Control Systems : Set 12 : Statespace (3) - Solutions

Prob 1 | The linearized equations of motion of the simple pendulum in the figure below are

$$\ddot{\theta} + \omega^2 \theta = u$$



a) Write the equations of motion in state-space form.

Defining $x_1 = \theta$ and $x_2 = \dot{\theta}$, and anticipating that the measured variable in part (2) is $\dot{\theta}$, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

b) Design an estimator (observer) that reconstructs the state of the pendulum given measurements of $\dot{\theta}$. Assume $\omega = 5$ rad/sec, and pick the estimator roots to be at $s = -10 \pm 10j$.

Characteristic equation is given by

$$\det(sI - A + LC) = 0$$

$$\det \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \right) = L_2 s - L_1 \omega^2 + \omega^2 + s^2 = 0$$

Equate to the desired characteristic equation, and solve

$$L_2 s - L_1 \omega^2 + \omega^2 + s^2 = s^2 + 20s + 200$$

The result is

$$L = \begin{bmatrix} -7 \\ 20 \end{bmatrix}$$

c) Write the transfer function of the estimator between the measured value of $\dot{\theta}$ and the estimated value of θ .

To find the transfer function from the measured value of $\dot{\theta}$, y , to the estimated value of θ , $\hat{\theta}$, we use the estimator equations

$$\begin{aligned}\hat{\dot{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) \\ &= (A - LC)\hat{x} + Bu + Ly\end{aligned}$$

We now convert this to a transfer function form to get the transfer function from the input to this system, y , to the output $\hat{\theta} = \hat{x}_1$

$$\begin{aligned}\frac{\hat{\Theta}(s)}{Y(s)} &= [1 \quad 0] (sI - A + LC)^{-1} L \\ &= -\frac{7s - 20}{s^2 + 20s + 200}\end{aligned}$$

- d) Design a controller (that is, determine the state feedback gain K) so that the roots of the closed-loop characteristic equation are at $s = -4 \pm 4j$.

For the controller gain $K = [K_1 \quad K_2]$, we require

$$\det(sI - A + BK) = 0 \quad \rightarrow \quad s^2 + K_2s + \omega^2 + K_1 = 0$$

The desired characteristic equation is

$$(s + 4 + 4j)(s + 4 - 4j) = s^2 + 8s + 32$$

Equating coefficients and solving gives $K = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

Prob 2 | A certain process has the transfer function

$$G(s) = \frac{4}{s^2 - 4}$$

a) Find A , B and C for this system in observer canonical form

The observer canonical form can be read directly from the transfer function

$$A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \quad C = [1 \quad 0]$$

b) If $u = -Kx$, compute K so that the closed-loop control poles are located at $s = -2 \pm 2j$

The desired characteristic equation is

$$\alpha(s) = (s + 2 - 2j)(s + 2 + 2j) = s^2 + 4s + 8$$

and the characteristic equation of the closed-loop system

$$\det(sI - A + BK) = s^2 + 4 * K_2 * s + 4 * K_1 - 4$$

Equating coefficients and solving gives $K = [3 \quad 1]$

c) Compute L so that the estimator-error poles are located at $s = -10 \pm 10j$

The desired characteristic equation is

$$\alpha_e(s) = (s + 10 + 10j)(s + 10 - 10j) = s^2 + 20s + 200$$

and the characteristic equation of the estimator is

$$\det(sI - A + LC) = s^2 + L_1s + L_2 - 4$$

Equating coefficients and solving gives $L = \begin{bmatrix} 20 \\ 204 \end{bmatrix}$

d) Give the transfer function of the resulting controller

The transfer function is

$$\begin{aligned} K(s) &= -K(sI - A + LC + BK)^{-1}L \\ &= -[3 \quad 1] \begin{bmatrix} s + 20 & -1 \\ 212 & s + 4 \end{bmatrix}^{-1} \begin{bmatrix} 20 \\ 204 \end{bmatrix} \\ &= \frac{-264s - 692}{s^2 + 24s + 292} \end{aligned}$$

To check this answer, one can use the ss2tf command in Matlab

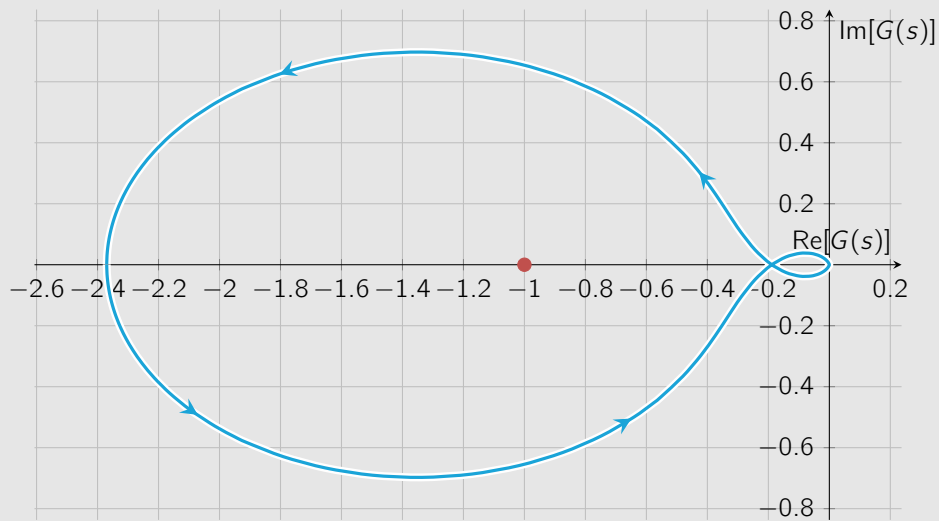
- e) What are the gain and phase margins of the controller and the given open-loop system?

The loop gain is given by

$$K(s)G(s) = \frac{264s + 692}{s^2 + 24s + 292} \cdot \frac{4}{s^2 - 4}$$

We can see the Nyquist plot for this system below from which we notice that the system has both a positive and negative gain margin - the gain can be increased by 5 times, or decreased by 0.41 times before the system becomes unstable.

From the plot, we can also see that the phase margin is about 35°



Prob 3 | Consider the control of

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{s(s+1)}$$

- a) Let $y = x_1$ and $\dot{x}_1 = x_2$ and write the state equations for the system

Write the ODE

$$\ddot{y} + \dot{y} = u$$

Define the state variables $x_1 = y$, $x_2 = \dot{x}_1 = \dot{y}$ and we can give the state equations as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

- b) Find K_1 and K_2 so that $u = -K_1x_1 - K_2x_2$ yields closed-loop poles with a natural frequency $\omega_n = 3$ and a damping ratio $\zeta = 0.5$

Desired characteristic equation

$$\begin{aligned} \alpha_c(s) &= s^2 + 2\zeta\omega_n s + \omega_n^2 \\ &= s^2 + 3s + 9 \end{aligned}$$

Closed-loop characteristic equation

$$\det(sI - A + BK) = 10K_1 + s + 10K_2s + s^2$$

Equating coefficients and solving gives $K = \begin{bmatrix} \frac{9}{10} & \frac{1}{5} \end{bmatrix}$

- c) Design a state estimator for the system that yields estimator error poles with $\omega_{n1} = 15$ and $\zeta_1 = 0.5$

Running the code below, we get that $L = \begin{bmatrix} 14 \\ 211 \end{bmatrix}$

```
wn = 15
zeta = 0.5
A = [0 1; 0 -1];
B = [0; 10];
C = [1 0];
L = place(A', C', roots([1 2*zeta*wn wn^2]))'
```

- d) What is the transfer function of the controller obtained by combining parts (a) - (c)?

```
% Compute transfer function
ctrl = ss(A-L*C-B*K, L, -K, 0)
tf(ctrl)
```

```
ans =
```

$$\frac{-54.8 s - 202.5}{s^2 + 17 s + 262}$$

Prob 4 | The linearized longitudinal motion of a helicopter near hover (figure below) can be modeled by the normalized third-order system

$$\begin{bmatrix} \dot{q} \\ \dot{\theta} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} -0.4 & 0 & -0.01 \\ 1 & 0 & 0 \\ -1.4 & 9.8 & -0.02 \end{bmatrix} \begin{bmatrix} q \\ \theta \\ u \end{bmatrix} + \begin{bmatrix} 6.3 \\ 0 \\ 9.8 \end{bmatrix} \delta$$

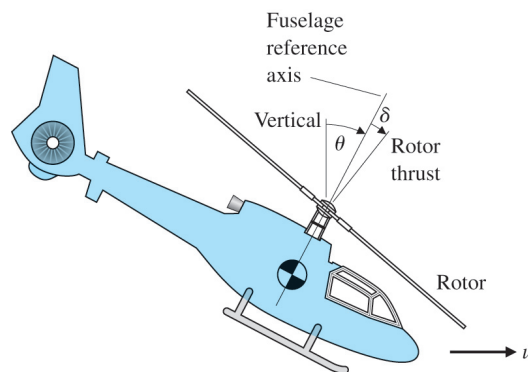
where

q = pitch rate

θ = pitch angle of fuselage

u = horizontal velocity (standard aircraft notation)

δ = rotor tilt angle (control variable)



Suppose our sensor measures the horizontal velocity u as the output, that is $y = u$. Use Matlab to answer the following questions.

- Design a state estimator with poles at -8 and $-4 \pm 4\sqrt{3}j$
- Compute the compensator transfer function using control you design for this problem in exercise set 5, and the estimator designed above.
- Draw Bode plots for the loop gain and the closed-loop system. What is the bandwidth, gain margin and phase margin?

```
clear

A = [-0.4 0 -0.01; 1 0 0; -1.4 9.8 -0.02];
B = [6.3; 0; 9.8];
C = [0 0 1];

G = ss(A,B,C,0);

% Design estimator
p = [-8, -4+4*sqrt(3)*j, -4-4*sqrt(3)*j];
L = place(A', C', p)'
```

```

15 L =
    44.71
    18.813
    15.58

20 eig(A-L*C) % Check result

25 ans =
    -8 + 0i
    -4 + 6.9282i
    -4 - 6.9282i

    % Control from previous exercise
    K = [0.47 1 0.063];

30 % Compute the compensator
    ctrl = tf(ss(A-L*C-B*K, L, K, 0))

    ctrl =
35
    40.81 s^2 + 61.02 s + 32.01
    -----
    s^3 + 19.58 s^2 - 210.2 s + 814

    % Bode plot - loop gain
40 figure(1)
    margin(ctrl*G)
    [Gm,Pm,Wcg,Wcp] = margin(ctrl*G)

45 Gm =
    0.73889

    Pm =
50
    22.022

    Wcg =
55
    6.9471

    Wcp =
60
    11.721

    % Close-loop bode plot

```

```
figure(2)
65 bode(feedback(ctrl*G, 1))

bandwidth(feedback(ctrl*G, 1))

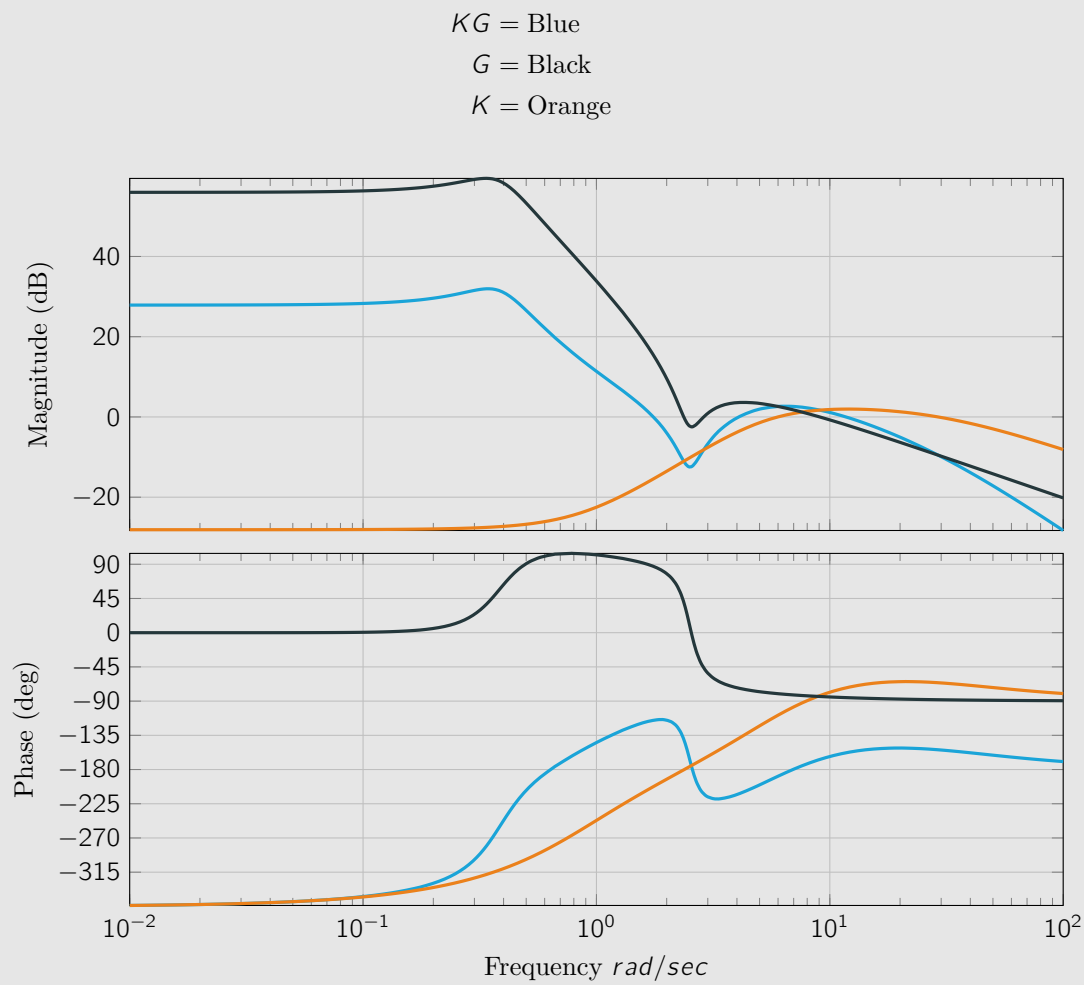
70 ans =

      2.0007

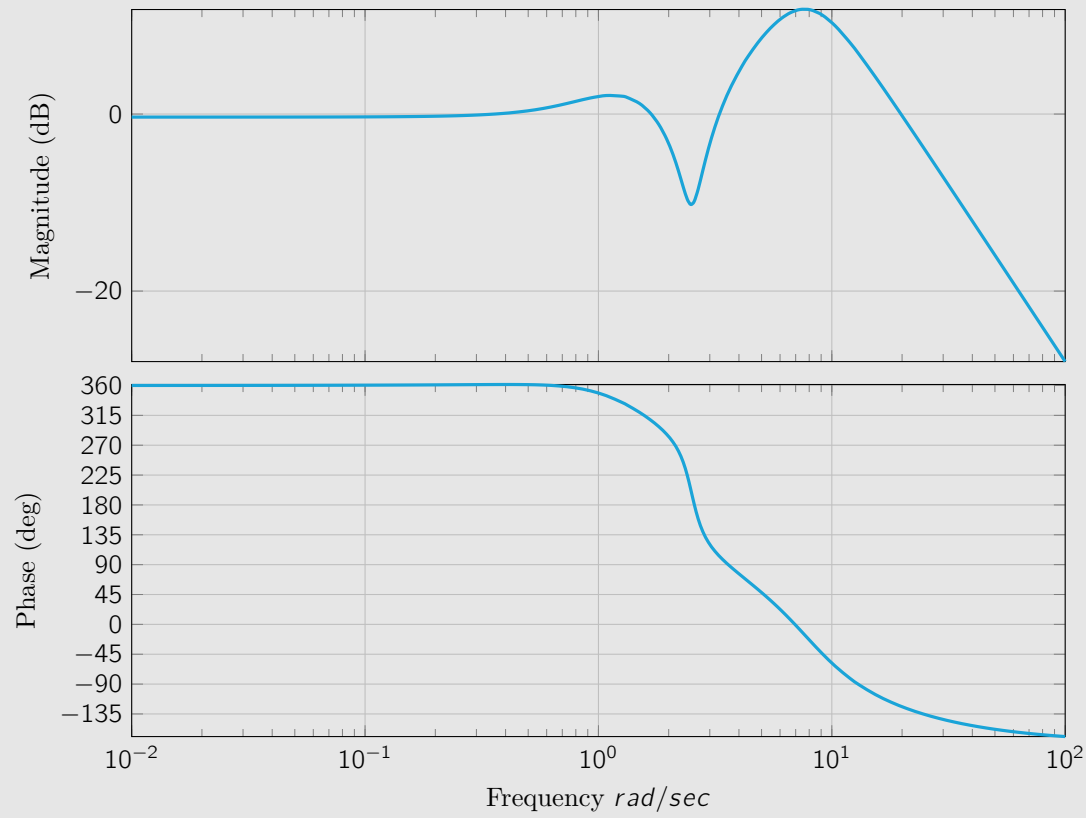
% Controller
```



Bode plot of loop gain



Closed-loop bode plot



Prob 5 | The linearized equations of motion for a satellite are

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The inputs u_1 and u_2 are the radial and tangential thrusts, the state variables x_1 and x_3 are the radial and angular deviations from the reference (circular) orbit, and the outputs y_1 and y_2 are the radial and angular measurements, respectively.

a) Show that the system is controllable using both control inputs

Compute the controllability matrix

$$\mathcal{C} = [B \quad AB \quad A^2BA^3B] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2\omega & -\omega^2 & 0 \\ 1 & 0 & 0 & 2\omega & -\omega^2 & 0 & 0 & -2\omega^3 \\ 0 & 0 & 0 & 1 & -2\omega & 0 & 0 & -4\omega^2 \\ 0 & 1 & -2\omega & 0 & 0 & -4\omega^2 & 2\omega^3 & 0 \end{bmatrix}$$

Looking at the first four columns, we note that we have a full rank matrix, and so the system is controllable.

b) Show that the system is controllable using only a single input. Which one is it?

Compute the controllability matrix for each input / column of the matrix B .

First input : $B = [0 \quad 1 \quad 0 \quad 0]^T$

$$\mathcal{C} = \begin{bmatrix} 0 & 1 & 0 & -\omega^2 \\ 1 & 0 & -\omega^2 & 0 \\ 0 & 0 & -2\omega & 0 \\ 0 & -2\omega & 0 & 2\omega^3 \end{bmatrix}$$

Computing the determinant of this matrix gives zero, which tells us that the system is uncontrollable.

Second input : $B = [0 \quad 0 \quad 0 \quad 1]^T$

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 2\omega & 0 \\ 0 & 2\omega & 0 & -2\omega^3 \\ 0 & 1 & 0 & -4\omega^2 \\ 1 & 0 & -4\omega^2 & 0 \end{bmatrix}$$

Computing the determinant gives $-12\omega^4$, which tells us that the controllability matrix is full rank for all ω , and therefore the system is controllable.

- c) Show that the system is observable using both measurements.

Compute the observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & -2\omega & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 \\ -6\omega^3 & 0 & 0 & -4\omega^2 \end{bmatrix}$$

Where we notice from the first four rows that this system is observable.

- d) Show that the system is observable using only one measurement. Which one it it?

Compute the observability matrix for each output / row of the matrix C .

First output $C = [1 \ 0 \ 0 \ 0]$

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & -\omega^2 & 0 & 0 \end{bmatrix}$$

We see from the third column that this matrix is not full rank, and therefore the system is not observable.

Second output $C = [0 \ 0 \ 1 \ 0]$

$$\mathcal{O} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \\ -6\omega^3 & 0 & 0 & -4\omega^2 \end{bmatrix}$$

Calculating the determinant, we get $-12\omega^4$, which tells us that the matrix is full-rank for all non-zero ω .

Note that the definitions of controllability and observability matrices are valid for multiple inputs and outputs, and that full rank of these matrices is what's required for controllability and observability.

Prob 6 | Consider a system with the transfer function

$$G(s) = \frac{9}{s^2 - 9}$$

a) Find (A, B, C) for this system in observer canonical form

For a transfer function

$$G(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2}$$

the observer canonical form becomes

$$A = \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

b) Is (A, B) controllable?

Check the rank of the controllability matrix

$$\mathcal{C} = [B \quad AB] = \begin{bmatrix} 0 & 9 \\ 9 & 0 \end{bmatrix}$$

The determinant is -81 , and so the system is controllable.

c) Compute K so that the closed-loop poles are assigned to $s = -3 \pm 3j$

We use Matlab's place function to calculate $K = \begin{bmatrix} 3 & \frac{2}{3} \end{bmatrix}$

d) Is the system observable?

The system is in observer canonical form, and therefore it is observable.

e) Design an estimator with estimator poles at $s = -12 \pm 12j$

$$\det(sI - A + LC) = s^2 + L_1 s + L_2 - 9 = s^2 + 24s + 288$$

$$\text{Gives } L = \begin{bmatrix} 24 \\ 297 \end{bmatrix}$$

f) Suppose the system is modified to have a zero

$$G_1(s) = \frac{9(s+1)}{s^2 - 9}$$

Prove that if $u = -Kx + r$, there is a feedback gain K that makes the closed-loop system unobservable. (Again assume an observer canonical realization for $G_1(s)$.)

The observer canonical form is

$$A = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 9 \\ 9 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Given a state-feedback controller of the form $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$, the closed-loop observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ C(A - BK) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -9K_1 & 1 - 9K_2 \end{bmatrix}$$

which has a determinant of $1 - 9K_2$. We can therefore make the closed-loop system unobservable by choosing $K_2 = \frac{1}{9}$.

The poles of the closed-loop system for $K_2 = \frac{1}{9}$ are given by

$$\det(sI - A + BK) = \det \begin{bmatrix} 9K_1 + s & 0 \\ 9K_1 - 9 & s + 1 \end{bmatrix} = (9K_1 + s)(s + 1)$$

So we see that the system has become unobservable due to a pole-zero cancellation.